1908MNRAS..68..496T

Garrett, The Jaipur Observatory and its builder, presented by the author; Greenwich Astrographic Catalogue, vol. ii., 16 charts of the Astrographic Chart of the heavens, and Observations of the planet Eros, 1900–1901, presented by the Royal Observatory, Greenwich; Professor E. S. Holden, Galileo, and other tracts, presented by the author; Oxford Astrographic Catalogue, vol. iv., presented by the University Observatory, Oxford; Pennsylvania University Publications, Catalogue of 648 double stars discovered by Professor Hough, presented by the University; three lantern slides of the Corona of 1908 January 3, taken by Professor Campbell, presented by the Lick Observatory.

## *An Empirical Law of Astronomical Refraction.*

By H. H. Turner, D.Sc., F.R.S., Savilian Professor.

1. The following investigation was originally undertaken with the view of substituting a simple proof of the law of refraction for students who could not follow the more elaborate proof involving the differential equation. But it was a surprise to find how closely the observed refraction could be represented with so rough a supposition as that of three, or even two, homogeneous shells of atmosphere; and the question was suggested whether, in the present state of our knowledge, more elaborate hypotheses were really justified. If a rough supposition fits the facts, clearly it is no proof of the correctness of a more elaborate one that it also fits the facts.

2. Moreover, suspicion of the correctness of existing hypotheses was suggested from another direction. Meteorologists are finding that the temperature of the atmosphere does not follow a smooth gradient, as is generally assumed in refraction hypotheses: at a certain height a wholly unexpected state of things has been found to exist. According to M. Teisserenc de Bort there is above 10 or 12 km., an "isothermal layer" in which the temperature ceases to fall as we ascend; and the conditions are similar over parts of the world where the temperatures close to the surface differ widely. It is difficult to reconcile these results of observation with the hypotheses usually adopted in constructing tables of refraction.* It is, however, not intended to examine at present the consequences of taking M. Teisserenc de Bort's work into account;—merely to show that it may not be difficult to do so when we have fuller information, without, perhaps, dislocating existing refraction tables.

3. First let us consider an atmosphere of one homogeneous spherical layer. Let C be the Earth's centre; OM a section of its surface through C and the star; LB the section of the boundary of

---

* I am indebted to Mr. Saunder for a reference to a paper by Professor Bakhuyzen in *Konink. Akad. van Weten. te Amsterdam*, 1907 January 26; see *Nature*, 1907 April 4, p. 538; in which the discordance between observation and the assumption usually made is demonstrated by a concrete example.

1908MNRAS..68..496T

the atmosphere; ABO the direction of the star's light, refracted once only at the boundary B. The star is thus seen in the direction OBD, instead of in the direction BA.

If $\angle ABE = \phi, \quad \angle OBC = \psi, \quad \angle BOL = \zeta,$

then by the law of refraction

$$\sin \phi = \mu \sin \psi \quad . \quad . \quad . \quad . \quad (1)$$

and from the triangle OBC

$$\sin \psi = \frac{OC}{BC} \sin \zeta = \frac{1}{1+z} \sin \zeta \quad . \quad . \quad . \quad (2)$$

if $z$ be the ratio of the height of the atmosphere to the Earth's radius.

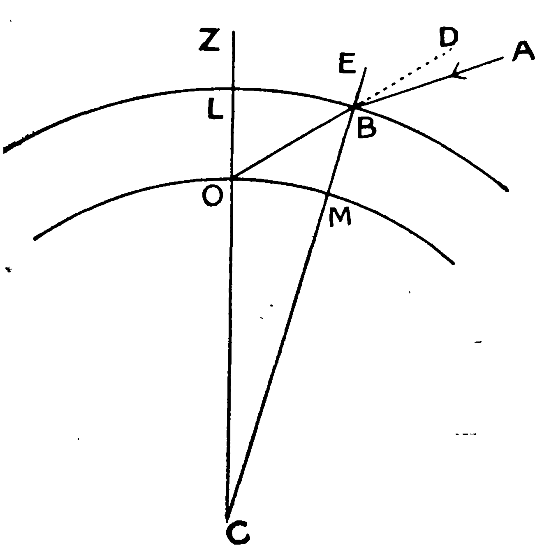

FIG. 1.

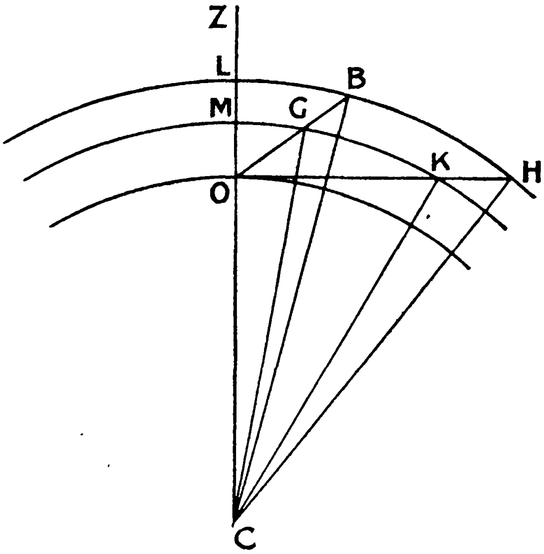

FIG. 2.

Hence $$\sin \phi = \frac{\mu}{1+z} \sin \zeta \quad . \quad . \quad . \quad . \quad (3)$$

4. The similarity of equations (2) and (3) suggests a geometrical construction for the refraction which is obvious enough, but which I do not remember to have seen in print.

With radius $\frac{1+z}{\mu}$ describe the circle MGK round C as centre (fig. 2).

Then $$\frac{\sin OGC}{\sin GOL} = \frac{1}{CG} = \frac{\mu}{1+z}.$$

Hence OGC $= \phi$, just as OBC $= \psi$, and the refraction is $\phi - \psi =$ GCB.

5. If now we take the case of two homogeneous shells of atmosphere, we see that the refraction will be represented by the sum

1908MNRAS..68..496T

of two angles $G_1CB_1$, $G_2CB_2$, with a gap between; the radii $CG_1$, $CB_1$, etc. being of constant length, and the angle ZOG being $\zeta$, the apparent zenith distance, and generally, if there are $n$ layers, expressing all the angles of incidence and refraction in terms of $\zeta$, we have—

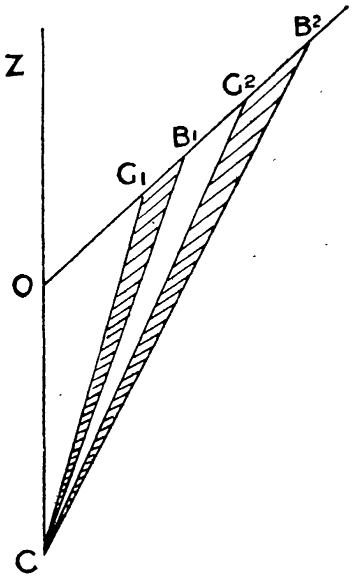

FIG. 3.

$$\left.\begin{aligned}
\sin\psi_1 &= \frac{1}{1+z_1}\sin\zeta = \frac{\mu_1}{\mu_1(1+z_1)}\sin\zeta \\
\sin\phi_1 &= \frac{\mu_1}{\mu_2}\sin\psi_1 = \frac{\mu_1}{\mu_2(1+z_1)}\sin\zeta \\
\sin\psi_2 &= \frac{1+z_1}{1+z_2}\sin\phi_1 = \frac{\mu_1}{\mu_2(1+z_2)}\sin\zeta \\
\sin\phi_2 &= \frac{\mu_2}{\mu_3}\sin\psi_2 = \frac{\mu_1}{\mu_3(1+z_2)}\sin\zeta \\
&\vdots \\
\sin\psi_n &= \frac{\mu_1}{\mu_n(1+z_n)}\sin\zeta \\
\sin\phi_n &= \frac{\mu_1}{\mu_{n+1}(1+z_n)}\sin\zeta
\end{aligned}\right\} \quad . \quad (4)$$

The notation is tolerably obvious: the radii of the successive surfaces of separation are $1+z_1$, $1+z_2$, $1+z_3 \cdots 1+z_n$; and the refractive indices of the successive homogeneous shells are $\mu_1$, $\mu_2 \cdots \mu_n$. If the $n^{\text{th}}$ be the outermost, $\mu_{n+1} = 1$.

1908MNRAS..68..496T

6. It will be seen that although $B_1$ (fig. 3) lies on the boundary of the first shell, $B_2$ does not lie on the boundary of the second. The radius $CB_2$ is not $1+z_2$ but $\frac{\mu_2}{\mu_1}(1+z_2)$; and if the refractive index decreases with the height, this is less than $1+z_2$.

Recent meteorological experiments, however, suggest that there may be anomalies in the upper strata. If the refractive index for the $r^{\text{th}}$ layer should increase with height instead of decreasing, then $CG_r$ would be greater than $CB_r$, and the angle $G_rCB_r$ would be subtractive.

7. It is of interest to consider whether there are always gaps between the elements. If the angle $B_1CG_2$ vanishes, then $\phi_2=\psi_1$, or

$$\mu_1(1+z_1)=\mu_3(1+z_2) \quad . \quad . \quad . \quad . \quad (5)$$

In this case the refractions of the two assumed layers are exactly equivalent, at all zenith distances, to that of a single layer with outer radius $CB_2=\frac{\mu_2}{\mu_1}(1+z_2)$, and of refractive index $\frac{CB_2}{CG_1}=\frac{1+z_2}{1+z_1}$.

8. Condition (5) may be regarded as an equation determining $\mu_3$, the refractive index of the shell surrounding the two inner ones; and hence we see that, *given any two homogeneous spherical shells of atmosphere with any refractive indices, they may be replaced for purposes of astronomical refraction by a single homogeneous shell, provided the refractive index of the medium immediately surrounding them has a certain value, and in that case only.*

This case must be carefully distinguished from the one in which a really homogeneous shell is arbitrarily treated as two by an imaginary surface of separation, at which there is no refraction. In this latter case one of the angles $G_1CB_1$ or $G_2CB_2$ would be zero; not necessarily the gap $B_1CG_2$.

9. Attention is drawn to the possibility above italicised, because it seemed at first that it might afford the explanation of the curious fact that the rough hypothesis of two or three homogeneous shells of air can give so good an approximation to our observed refractions as below. If it is possible to replace two elements by a single one, and to continue the process, there is nothing unreasonable in the exact representation of a large number of elements by a simple equivalent. But the numerical values obtained do not support this view, the gaps being wide compared with the thickness of the shells.

10. We now turn to numerical evaluation of the constants. The law of refraction is simple enough at moderate zenith distances, and we shall be chiefly concerned with its form near the horizon. In that case $\phi$, $\psi$, and $\zeta$ are all nearly 90°, and it is more convenient to deal with their complements, which are small angles. Let us therefore write

$$\zeta=90^\circ-A, \quad \phi=90^\circ-\nu, \quad \psi=90^\circ-\lambda.$$

1908MNRAS..68..496T

Let $\alpha$ and $\beta$ be the values of $\lambda$ and $\nu$ for $\zeta = 90°$ or $A = 0°$. Then from equations (2) and (3),

$$\cos\alpha = \frac{1}{1+z}, \quad \cos\beta = \frac{\mu}{1+z} \quad . \quad . \quad . \quad (6)$$

and equations (2) and (3) may now be written,

$$\cos\lambda = \cos\alpha\cos A, \quad \cos\nu = \cos\beta\cos A \quad . \quad . \quad (7)$$

In fig. 2, $\angle OCH = \alpha$ and $\angle OCK = \beta$ : $\angle KCH = \alpha - \beta$. The refraction is $r = \phi - \psi = \lambda - \nu$.

11. In terms of the method usually adopted, we should find our constants from what we could learn of $\mu$ and $z$. From the point of view of this note, we are to determine $\alpha$ and $\beta$ empirically from observed refractions at two different zenith distances; and the most convenient Z.D.'s to select are 45° and 90°. We shall confine our attention to mean refractions; and it is convenient to have the horizontal refraction exactly 33′, for a reason which will become clear when we deal with three separate layers. The mean Pulkowa refractions, as shown in the second column of Table I., have therefore been multiplied by a constant. Hence one equation between $\alpha$ and $\beta$ is

$$\alpha - \beta = 33' = \cdot 00960 \quad . \quad . \quad . \quad (8)$$

12. At altitude 45° the equations may be combined to obtain the usual approximate law of tan Z.D. Subtracting one of equations (8) from the other, we have

$$(\cos\nu - \cos\lambda) = (\cos\beta - \cos\alpha)\cos A,$$

or

$$2\sin\frac{\lambda-\nu}{2} = (\cos\beta - \cos\alpha)\cos A \operatorname{cosec}\frac{\lambda+\nu}{2}.$$

As a first approximation, we can neglect the differences between $\lambda$, $\nu$, and A on the right, and put $\frac{\lambda+\nu}{2} = A = 90° - \zeta$. If the refraction be denoted by $r$, we have for moderate zenith distances

$$r = 2\sin\frac{\lambda-\nu}{2}.$$

$$\therefore \quad r = (\cos\beta - \cos\alpha)\tan\zeta.$$

So that $(\cos\beta - \cos\alpha)$ is the well-known "coefficient of refraction" at moderate zenith distances: and when $\zeta = 45°$, $\tan\zeta = 1$, and

$$\cos\beta - \cos\alpha = 55''\cdot 70 = \cdot 0002700 \quad . \quad . \quad (9)$$

Hence

$$2\sin\frac{\alpha-\beta}{2}\sin\frac{\alpha+\beta}{2} = \cdot 0002700.$$

1908MNRAS..68..496T

13. Substituting the value of $\alpha-\beta$ from equation (9), we find

$$\tfrac{1}{2}(\alpha+\beta)=1^\circ\ 36'\ 43'',$$

and since

$$\tfrac{1}{2}(\alpha-\beta)=\quad 16\ \ 30$$

$$\alpha=1^\circ\ 53'\ 13'' \qquad \beta=1^\circ\ 20'\ 13'' \quad . \quad . \quad (10)$$

14. We can now estimate the error of the approximation above, where A is substituted for $\frac{\lambda+\nu}{2}$. At $A=45^\circ$

$$\log\cos\nu=\log\cos 45^\circ-{\cdot}000118$$

$$\log\cos\lambda=\log\cos 45^\circ-{\cdot}000236$$

$$\therefore\quad \nu-45^\circ=56'', \quad \lambda-45^\circ=1'\ 52''.$$

These differences are found at once from a table of log cosines. Hence

$$\frac{\lambda+\nu}{2}=45^\circ\ 1'\ 24'',$$

and the accurate equation to replace equation (9) is

$$\cos\beta-\cos\alpha={\cdot}0002700\times 1{\cdot}00041$$

$$\tfrac{1}{2}(\alpha+\beta)=1^\circ\ 36'\ 43''+1''{\cdot}9$$

$$=1^\circ\ 36'\ 45''\ \text{say},$$

$$\therefore\quad \alpha=1^\circ\ 53'\ 15'' \quad \beta=1^\circ\ 20'\ 15'' \quad . \quad . \quad (11)$$

15. If a table of logarithms of cosines to seconds of arc is not available, we can throw the equation

$$\cos\lambda=\cos\alpha\ \cos A$$

into the form

$$\frac{\cos A-\cos\lambda}{\cos A+\cos\lambda}=\frac{1-\cos\alpha}{1+\cos\alpha},$$

or

$$\tan\frac{\lambda-A}{2}.\ \tan\frac{\lambda+A}{2}=\tan^2\frac{\alpha}{2}.$$

Let

$$\tan\frac{\lambda-A}{2}=\frac{\rho_1}{2}:\ \text{then since}\ \frac{\lambda+A}{2}=A+\frac{\lambda-A}{2}$$

$$\rho_1(\rho_1+2\tan A)=2\tan^2\frac{\alpha}{2}(2-\rho_1\tan A).$$

$$\therefore\quad \rho_1=-a\pm\sqrt{a^2+4b}$$

$$=\frac{4b}{a+\sqrt{a^2+4b}},$$ the upper sign being the only one to take; where $a=\tan A\sec^2\frac{\alpha}{2}$, $b=\tan^2\frac{\alpha}{2}$

$$\therefore\quad \rho_1=\frac{2(1-\cos\alpha)}{\tan A+\sqrt{\tan^2 A+\sin^2\alpha}} \quad . \quad . \quad (12)$$

1908MNRAS..68..496T

When we neglect sin $\alpha$ in comparison with tan A, the denominator becomes 2 tan A, and

$$\rho_1 = (1 - \cos\alpha)\cot A.$$

Similarly,

$$\rho_2 = (1 - \cos\beta)\cot A,$$

and $r = \rho_1 - \rho_2 = (\cos\beta - \cos\alpha)\cot A$ as before.

When sin $\alpha$ is small compared with tan A we have

$$\rho_1 = (1 - \cos\alpha)\cot A\left[1 - \frac{1}{4}\frac{\sin^2\alpha}{\tan^2 A}\right] \quad . \quad . \quad (13)$$

and when A = 45°

$$\rho_1 = (1 - \cos\alpha)(1 - \tfrac{1}{4}\sin^2\alpha)$$

$$= \frac{\alpha^2}{2} - \frac{\alpha^4}{6} \text{ to the second order in } \rho_1.$$

Similarly, $\rho_2 = \frac{\beta^2}{2} - \frac{\beta^4}{6}$; and to this order of approximation $\rho_1$ and $\rho_2$ are equal to their tangents. Hence the refraction is

$$\rho_1 - \rho_2 = (\alpha - \beta) \cdot \frac{\alpha+\beta}{2}\left[1 - \tfrac{1}{3}(\alpha^2 + \beta^2)\right] \quad . \quad . \quad (14)$$

Since $\alpha - \beta$ is the horizontal refraction we have

$$k = \frac{\text{Refraction at } 45^\circ}{\text{Refraction at Horizon}} = \frac{\alpha+\beta}{2}\left[1 - \tfrac{1}{3}(\alpha^2 + \beta^2)\right] \quad . \quad (15)$$

16. Let us now consider what happens if we have $n$ concentric shells characterised by

$$(\alpha_1\ \beta_1), (\alpha_2\beta_2) \ldots (\alpha_n, \beta_n).$$

In each case the horizontal refraction is $(\alpha_r - \beta_r)$; so that the total horizontal refraction is

$$(\alpha_1 - \beta_1) + (\alpha_2 - \beta_2) + (\alpha_3 - \beta_3) + \ldots + (\alpha_n - \beta_n) = 33'.$$

Now the division of our shells is somewhat arbitrary. We can make them all of equal thickness, for instance; or we can choose their relative thicknesses so that they have equal refractions at some specified zenith distance. There is a convenience in making their *horizontal refractions all equal*, so that

$$\alpha_1 - \beta_1 = \alpha_2 - \beta_2 = \alpha_3 - \beta_3 = \ldots = \alpha_n - \beta_n = \frac{1}{n}(33') \quad . \quad (16)$$

Proceeding to the refraction at 45°, it is

$$\sum(\alpha_r - \beta_r)\left(\frac{\alpha_r + \beta_r}{2}\right)\left[1 - \tfrac{1}{3}(\alpha_r^2 + \beta_r^2)\right]$$

and hence substituting for $\alpha_r - \beta_r$, we have

$$nk = \sum\left(\frac{\alpha_r + \beta_r}{2}\right)\left[1 - \tfrac{1}{3}(\alpha_r^2 + \beta_r^2)\right]. \quad . \quad . \quad (17)$$

1908MNRAS..68..496T

where $k$ denotes, as in equation (15), the ratio of refraction at 45° to that at the horizon.

## *One Shell.*

17. The numerical results for one shell (with $\alpha = 1^\circ\ 53'\ 15''$, $\beta = 1^\circ\ 20'\ 15''$, as in equation (11)) are shown in the third column of Table I., and the discordances from the Pulkowa refractions in the fourth column. It is clear that, though the accordance is close as far as Z.D. 60°, the hypothesis does not fit the facts at all. The discordance at Z.D. 89° is nearly 5′, or 20 per cent.

### Table I.

| Z.D. | Pulkowa Refraction. | Calculated one Shell. | O—C. | Calculated Two Shells. | | | |
|---|---|---|---|---|---|---|---|
| | | | | Inner. | Outer. | Sum. | O—C. |
| ° | ″ | ″ | ″ | ″ | ″ | ″ | ″ |
| 45 | 55·7 | 55·7 | 0·0 | 9·5 | 46·2 | 55·7 | 0·0 |
| 60 | 96·2 | 96·2 | 0·0 | 16·3 | 79·9 | 96·2 | 0·0 |
| 70 | 151·9 | 152·4 | − 0·5 | 26·0 | 126·0 | 152·0 | − 0·1 |
| 80 | 305·5 | 311·8 | − 6·3 | 53·5 | 253·5 | 307·0 | − 1·5 |
| 81 | 337·5 | 346·0 | − 8·5 | 59·7 | 280 | 339·7 | − 2·2 |
| 82 | 376·5 | 388 | − 11·5 | 66 | 312 | 378 | − 2 |
| 83 | 424·8 | 442 | − 17 | 77 | 352 | 429 | − 4 |
| 84 | 486·1 | 512 | − 24 | 89 | 402 | 491 | − 5 |
| 85 | 566·2 | 606 | − 40 | 107 | 466 | 573 | − 7 |
| 86 | 674·2 | 738 | − 64 | 133 | 549 | 682 | − 8 |
| 87 | 825·7 | 935 | − 109 | 177 | 654 | 831 | − 5 |
| 88 | 1049·0 | 1239 | − 190 | 260 | 792 | 1052 | − 3 |
| 89 | 1397·0 | 1678 | − 281 | 470 | 925 | 1395 | + 2 |
| 90 | 1980·0 | 1980 | 0 | 990 | 990 | 1980 | 0 |

In column for one shell,

$\alpha = 1^\circ\ 53'\ 15''$ $\qquad \log\cos\alpha = 9{\cdot}9997643$

$\beta = 1\ \ 20\ \ 15$ $\qquad \log\cos\beta = 9{\cdot}9998817$

In columns for two shells,

$\alpha_1 = 0^\circ\ 41'\ 0''$ $\qquad \log\cos\alpha_1 = 9{\cdot}9999691$

$\beta_1 = 0\ \ 24\ \ 30$ $\qquad \log\cos\beta_1 = 9{\cdot}9999890$

$\alpha_2 = 2\ \ 48\ \ 50$ $\qquad \log\cos\alpha_2 = 9{\cdot}9994760$

$\beta_2 = 2\ \ 32\ \ 20$ $\qquad \log\cos\beta_2 = 9{\cdot}9995735$

## *Two Shells.*

18. With two shells we have at first sight two more disposable constants; really only one. One of them is used up in the

1908MNRAS..68..496T

assumption that the horizontal refractions of the two shells are equal, as in § 16. Our four constants $\alpha_1$, $\beta_1$, $\alpha_2$, $\beta_2$ are therefore to be determined by the equations

$$\alpha_1 - \beta_1 = \alpha_2 - \beta_2 = 16\tfrac{1}{2}' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (18)$$

$$2k = \frac{\alpha_1 + \beta_1}{2}\left[1 - \tfrac{1}{3}(\alpha_1^2 + \beta_1^2)\right] + \frac{\alpha_2 + \beta_2}{2}\left[1 - \tfrac{1}{3}(\alpha_2^2 + \beta_2^2)\right]. \qquad (19)$$

and one more condition still to be imposed. Inspection of the discordances for one shell shown in Table I. suggests that we should reduce the largest discordance to zero, *i.e.* should satisfy the observed refraction at 89°. This can only be done by "trial and error" methods. It was found useful to form tables of cos $\alpha$ cos A for values of A = 1°, 2°, 3° . . . 10°; and for every minute of arc of $\alpha$. By their use, successive approximations gave the values

$$\alpha_1 = 0^\circ\ 41'\ 0'' \qquad \alpha_2 = 2^\circ\ 48'\ 30''$$
$$\beta_1 = 0\ \ 24\ \ 30 \qquad \beta_2 = 2\ \ 32\ \ 0$$

as a close approximation; the horizontal refraction of 33′ being divided into two of 16½′.

19. We must now examine how far these values satisfy the equation (19). They are close enough for substitution in the square terms. The following small table facilitates this substitution: the unit is ·00001.

TABLE II.

*Value of* $\tfrac{1}{3}\alpha^2$.

| | 10′. | 20′. | 30′. | 40′. | 50′. | 60′. |
|---|---|---|---|---|---|---|
| 0° | 1 | 1 | 2 | 3 | 6 | 10 |
| 1° | 14 | 18 | 23 | 28 | 34 | 41 |
| 2° | 48 | 56 | 64 | 73 | 82 | 91 |
| 3° | 101 | 112 | 124 | 136 | 149 | 162 |
| 4° | 176 | 190 | 205 | 221 | 237 | 254 |
| 5° | 271 | 289 | 307 | 326 | 346 | 366 |

Equation (19) runs

$$3^\circ\ 13'\ 25'' = 32'\ 45''\ (1 - \cdot 00005) + 2^\circ\ 40'\ 15''\ (1 - \cdot 00146)$$
$$= 3^\circ\ 13'\ 0'' - 0''\cdot 1 - 14''\cdot 0.$$

Hence our right-hand side is defective by 39″. Throwing this on to the second shell, which is less effective at small altitudes, we take as a trial

$$\alpha_1 = 0^\circ\ 41'\ 0'' \qquad \alpha_2 = 2^\circ\ 48'\ 50''$$
$$\beta_1 = 0\ \ 24\ \ 30 \qquad \beta_2 = 2\ \ 32\ \ 20$$

1908MNRAS..68..496T

With these values of the constants the refractions in columns 5 and 6 of Table I. were calculated; and it will be seen from column 8 that the error of their sum is small.

20. The question left outstanding is the choice of slightly different constants which shall perhaps satisfy the observations at (say) Z.D. 80°–87° rather better, at the expense of Z.D.'s 88°–90°, where observations are rare and possibly affected by errors of several seconds. In Table III. are given the effects of altering $\alpha_1$ and $\beta_1$ by 10′, and the similar effects of altering $\alpha_2$ and $\beta_2$ by 10′. It will be seen that we can, by increasing the constants for one layer, and diminishing those for the other by an exactly equal amount, leave the moderate Z.D.'s almost unaffected, and at the same time introduce sensible modifications into the large Z.D.'s.

TABLE III.

*Change for* 10′ *Alteration of Constants.*

| Z.D. | Inner Shell. | Outer Shell. | Difference. | Error from Table I. |
|---|---|---|---|---|
| ° | ″ | ″ | ″ | ″ |
| 45 | (3) | (3) | 0·015 | (0·0) |
| 60 | (5) | (5) | 0·065 | (0·0) |
| 70 | (8) | (8) | 0·25 | (0·0) |
| 80 | (16) | (15) | 2·2 | − 1·5 |
| 81 | (17) | (16) | 2·9 | − 2 |
| 82 | 20 | 18 | 2 | − 2 |
| 83 | 23 | 20 | 3 | − 2 |
| 84 | 26 | 22 | 4 | − 5 |
| 85 | 32 | 23 | 9 | − 7 |
| 86 | 40 | 24 | 16 | − 8 |
| 87 | 52 | 22 | 30 | − 5 |
| 88 | 74 | 18 | 56 | − 3 |
| 89 | 100 | 7 | 93 | + 2 |
| 90 | 0 | 0 | 0 | 0 |

21. The first five differences in the fourth column of Table III. were formed by a special method, as that used for the later ones is unsuitable. We may for moderate Z.D.'s simply differentiate the equation

$$\cos \lambda = \cos \alpha \cos \mathrm{A}.$$

Thus

$$\frac{d\lambda}{d\alpha} = \frac{\sin \alpha \cos \mathrm{A}}{(1 - \cos^2\alpha \, \cos^2\mathrm{A})^{\frac{1}{2}}}$$

$$= \frac{\sin \alpha}{(\tan^2\mathrm{A} + \sin^2\alpha)^{\frac{1}{2}}}.$$

The refraction due to a thin shell between $\alpha$ and $\alpha + \delta\alpha$ at altitude A

1908MNRAS..68..496T

is $\frac{d\lambda}{da} \cdot \delta a$. The change in this for a change $\Delta a$ in $a$ is $\frac{d^2\lambda}{da^2} \cdot \delta a \cdot \Delta a$: where

$$\frac{d^2\lambda}{da^2} = \frac{\tan^2 A \cos a}{(\sin^2 a + \tan^2 A)^{\frac{3}{2}}};$$

and when $a$ is small compared with A

$$\frac{d^2\lambda}{da^2} = \cot A \left[1 - \tfrac{1}{2}(1 + 3\cot^2 A)\sin^2 a\right].$$

If we have two shells with characteristics $a_1$ and $a_2$, the difference in effect will be proportional to

$$\tfrac{1}{2}\cot A\,(1 + 3\cot^2 A)\,(\sin^2 a_2 - \sin^2 a_1).$$

The value of the coefficient $\frac{1}{2}\cot A + \frac{3}{2}\cot^3 A$ is given in the following small table.

TABLE IV.

Value of $C = (\frac{1}{2}\cot A + \frac{3}{2}\cot^3 A)$ and $D = 0''{\cdot}00756\,C$.

| Z.D. | Alt. | C. | D. |
|---|---|---|---|
| ° | ° | | ″ |
| 45 | 45 | 2·00 | 0·015 |
| 60 | 30 | 8·6 | 0·065 |
| 70 | 20 | 32·6 | 0·25 |
| 80 | 10 | 286 | 2·2 |
| 81 | 9 | 380 | 2·9 |
| 82 | 8 | 545 | 4·1 |
| 83 | 7 | 813 | 6·1 |
| 84 | 6 | 1295 | 9·8 |
| 85 | 5 | 2240 | 16·9 |

The factor for the coefficient C in our special case is

$$10' \times \text{arc of } 16\tfrac{1}{2}' \times (\sin^2 2°\,40' \div \sin^2 0°\,33') = 0''{\cdot}00756.$$

The values of D for Z.D. 82°–85° are larger than those found directly in Table III., probably because the approximation used in Table IV. is ceasing to be accurate; but the quantities are of the same order of magnitude.

22. Table III. makes it clear that we cannot sensibly improve the accordance at 85° without introducing a much larger error at 89°. Every 1″ of alteration at 85° gives 10″ at 89°. Hence we may accept the constants used in Table I. as giving as good a representation of the Pulkowa refractions as we can fairly get with two shells of equal horizontal refraction.

1908MNRAS..68..496T

### *Three Shells.*

23. We divide the horizontal refraction equally into three of 11′. It is an obvious convenience to have whole minutes of arc, and for this reason the horizontal refraction was taken as exactly 33′ in the first instance. We might have taken 36′, which divides by 2, 3, and 4; but some work had already been done with 33′ before it was found that two shells would give such a good approximation.

After some few trials, the following values were found to be near the truth:—

$$\begin{array}{lll} \alpha_1 = 0^\circ\ 29'\ 0'' & \alpha_2 = 1^\circ\ 24'\ 0'' & \alpha_3 = 3^\circ\ 14'\ 0'' \\ \beta_1 = 0\ \ 18\ \ 0 & \beta_2 = 1\ \ 13\ \ 0 & \beta_3 = 3\ \ 3\ \ 0 \\ \hline \tfrac{1}{2}(\alpha_1+\beta_1) = 0\ \ 23\ \ 30 & \tfrac{1}{2}(\alpha_2+\beta_2) = 1\ \ 18\ \ 30 & \tfrac{1}{2}(\alpha_3+\beta_3) = 3\ \ 8\ \ 30 \end{array}$$

Hence equation (17) becomes in this case

$$4^\circ\ 50'\ 8'' = 23'\ 30''\ (1 - \cdot00003) + 1^\circ\ 18'\ 30''\ (1 - \cdot00034) + 3^\circ\ 8'\ 30''\ (1 - \cdot00191)$$

$$= 4^\circ\ 50'\ 30'' - 0''\cdot04 - 1''\cdot5 - 21''\cdot6$$

showing that the adjustment has been properly made.

TABLE V.

*Three Homogeneous Shells.*

| Z.D. | Inmost Shell. | Middle Shell. | Outmost Shell. | Sum = C. | O—C. | Difference for 10′. Inmost. | Middle. | Outmost. |
|---|---|---|---|---|---|---|---|---|
| ° | ″ | ″ | ″ | ″ | ″ | ″ | ″ | ″ |
| 45 | 4·47 | 15·09 | 36·14 | 55·70 | 0·00 | 1·95 | 1·85 | 1·85 |
| 60 | 7·67 | 26·16 | 62·42 | 96·25 | 0·00 | 3·56 | 3·18 | 3·18 |
| 70 | 12·21 | 41·34 | 98·27 | 151·82 | +0·1 | 5·30 | 5·05 | 5·05 |
| 80 | 25·3 | 85·0 | 196·0 | 306·3 | −0·8 | 11·3 | 10·2 | 9·1 |
| 81 | 28·3 | 94·4 | 215·8 | 338·5 | −1·0 | 12·2 | 11·1 | 10·2 |
| 82 | 31·7 | 106·0 | 239·7 | 377·4 | −0·9 | 13·9 | 12·5 | 10·9 |
| 83 | 36·2 | 120·7 | 269·0 | 425·9 | −1·1 | 16·2 | 14·4 | 11·6 |
| 84 | 42·4 | 140·5 | 305·5 | 488·4 | −2·3 | 18·5 | 16·1 | 11·8 |
| 85 | 51·0 | 167·0 | 350·5 | 568·5 | −2·3 | 22·3 | 18·6 | 12·5 |
| 86 | 63·5 | 205·5 | 407·0 | 676·0 | −1·8 | 27·7 | 21·8 | 13·2 |
| 87 | 85·0 | 264·0 | 477·5 | 826·5 | −0·8 | 39·0 | 28·5 | 11·5 |
| 88 | 125·5 | 362·0 | 556·5 | 1044·0 | +5·0 | 52·3 | 28·6 | 8·2 |
| 89 | 236·5 | 525·0 | 629·0 | 1390·5 | +6·5 | 83·2 | 19·5 | 2·3 |
| 90 | 660·0 | 660·0 | 660·0 | 1980·0 | 0·0 | 0·0 | 0·0 | 0·0 |

24. The results of this hypothesis are shown in Table V., and from the seventh column, O—C, it will be seen that there is fair accordance with the Pulkowa refractions. This accordance might be improved; by use of the differences given in the last

1908MNRAS..68..496T

three columns we could form linear equations to find small corrections to the mean radii of the three shells. But the improvement would be of the nature of a compromise, and would involve—

(*a*) A study of the accuracy of our knowledge of refractions at small altitudes. We should have to settle, for instance, what error at 89° Z.D. can be made to obtain an improvement of 1″ at Z.D. 80°.

(*b*) A more accurate computation of the differences for 10′. Those given above were found in the way made clear by the following example;—

| | 1st diff. | 2nd diff. |
|---|---|---|
| $\log\cos 18' \log\cos 10° = 9\cdot9933455 = \log\cos(10° + 16''\cdot0)$ | | |
| | 25″·3 | |
| $\log\cos 29' \log\cos 10° = 9\cdot9933361 = \log\cos(10° + 41''\cdot3)$ | | 12″·4 |
| | 37″·7 | |
| $\log\cos 40' \log\cos 10° = 9\cdot9933221 = \log\cos(10° + 79''\cdot0)$ | | |

The first two lines are formed to find the refraction due to the inmost layer (with constants 18′ and 29′) for altitude 10°, the result being 25″·3 as given in the 2nd column. By adding the 3rd line we find that the refraction of a shell with constants 29′ and 40′ (each 11′ greater than the former) is 37″·7; so that the difference for 11′ is 12″·4; and for 10′ would be 11″·3, which is given in the 7th column. But in forming differences of differences slight inaccuracies are multiplied, and inspection of the figures near the top of the last three columns shows that (since the numbers are nearly equal for all three shells) we are really concerned with differences of these again.

For the present, no attempt will be made to improve the approximation, and we turn to some other points.

25. It is of some interest to deduce the heights and refractive indices of the shells above found empirically.

### *One Shell.*

$$\log(1+z) = \log\sec\alpha = 0\cdot0002357 = \log 1\cdot000544$$

$$\log\mu = \log\cos\beta - \log\cos\alpha = 0\cdot0001174 = \log 1\cdot000270.$$

The "height of the homogeneous atmosphere" is thus indicated as 4000 × ·000544 miles, or 2·18 miles, which is a good deal smaller than that assigned by total pressure.

This is not surprising, for in replacing a series of successive bounding surfaces by a single surface, we should expect the equivalent single surface to lie in the midst of the constituents. We can imagine, for instance, a uniform graduation which would bring the equivalent surface to the mean height, *i.e.* would give us a "height of the homogeneous atmosphere" just half that assigned by calculating the total pressure; and it would not be surprising to find that the equivalent surface should be lower than this.

1908MNRAS..68..496T

### *Two Shells.*

Equations (4) of § 5 may be put in the form

| | Log. | Number. |
|---|---|---|
| $\log(1+z_1)=\log\ \mu_1/\mu_1+\log\sec\alpha_1=$ | 0·0000309 | 1·000071 |
| $\log\mu_1/\mu_2=\log(1+z_1)+\log\cos\beta_1=$ | 0·0000199 | 1·000046 |
| $\log(1+z_2)=\log\ \mu_1/\mu_2+\log\sec\alpha_2=$ | 0·0005439 | 1·001253 |
| $\log\mu_1/\mu_3=\log(1+z_2)+\log\cos\beta_2=$ | 0·0001174 | 1·000270 |

If we put $\mu_3=1$, then $\mu_1=1{\cdot}000270$, $\mu_2=1{\cdot}000224$. The heights of the bounding surfaces in miles are $z_1=0{\cdot}28$, $z_2=5{\cdot}00$.

### *Three Shells.*

The method of formation of the quantities has been made clear in the last case.

| | | | Log. | Number. |
|---|---|---|---|---|
| | $\log(1+z_1)=0+$ | $\log\sec\alpha_1=$ | 0·0000154 | 1·000035 |
| For | $\log\mu_1/\mu_2$, add | $\log\cos\beta_1$, | 0·0000094 | 1·000022 |
| ,, | $\log(1+z_2)$ ,, | $\log\sec\alpha_2$, | 0·0001391 | 1·000321 |
| ,, | $\log\mu_1/\mu_3$ ,, | $\log\cos\beta_2$, | 0·0000412 | 1·000097 |
| ,, | $\log(1+z_3)$ ,, | $\log\sec\alpha_3$, | 0·0007331 | 1·001689 |
| ,, | $\log\mu_1/\mu_4$ ,, | $\log\cos\beta_3$, | 0·0001175 | 1·000270 |

If we put $\mu_4=1$, we have $\mu_1=1{\cdot}000270$, as in equation (9). This gives a check on the work.

Further, $\mu_2=1{\cdot}000248$, $\mu_3=1{\cdot}000173$. The heights of the bounding surfaces are, in miles,

$$z_1=0{\cdot}14,\quad z_2=1{\cdot}28,\quad z_3=6{\cdot}76.$$

It does not seem likely that these figures can have any physical interpretation. They merely emphasise the fact that a serious part of the refraction takes place near the Earth, and not high in the atmosphere. By comparing the columns for the separate shells shown in Tables I. and V. it will be seen how rapid is the increase in importance of the inner shells as we approach the horizon. In Table V., for instance, the two inner shells together only give half the effect of the outmost at moderate Z.D.'s; and they do not produce so great an effect as the outmost until we reach Z.D. 88°, when they surpass and ultimately double it in the remaining 2° or 3°. This suggests that any influence of meteorological phenomena on astronomical refractions must be sought at large Z.D.'s, which is in accordance with experience. But discussion of such points in detail cannot be undertaken at present.

### *Summary.*

§§ 1–2. Introductory.

§§ 3–6. A graphical representation of the refraction of a homogeneous spherical shell of atmosphere and of several such shells.

1908MNRAS..68..496T

§§ 7–8. Condition that two or more such shells should be replaceable by an equivalent single shell.

§§ 10–15. Empirical determination of constants for a single shell from observed refractions at Z.D. 45° and at the horizon.

§ 16. Extension of formulæ to two or more shells.

§ 17. Numerical results for one shell. The errors at 80°, 85°, and 89° are respectively −6″, −40″, −281″. The hypothesis cannot be said to fit the facts at all beyond Z.D. 60° or 70° at most. See Table I.

§§ 18–22. Numerical results for two shells. See Table I. The errors at 80°, 85°, and 89° are reduced to −1″·5, −7″, and +2″—a very fair accordance for so obviously rough an assumption.

§§ 23–24. Numerical results for three shells: see Table V. The improvement is not great, much less marked than before; and it seems clear that the principal step was taken in passing from one shell to two.

§ 25. Some numerical values for heights, etc.

## *The Perturbations of Halley's Comet in the Past. Fourth Paper. The period* 760 *to* 1066.

By P. H. Cowell, M.A., F.R.S., and A. C. D. Crommelin, B.A.

We have again to acknowledge the kind assistance of Dr. Smart and Messrs. F. R. Cripps and Thos. Wright in these calculations.

In the last paper, *M.N.*, lxviii. 5, p. 378, we found March 27 for the approximate date of perihelion passage in 1066, and 44″·686 for the value of $n$ at that date. For the preceding passage we used the date given by Hind (989 September 12) for the purpose of computing the perturbations, and the following results indicate that this date is correct within a few days:—

*Revolution* 989–1066.

| Planet. | Limits of $u$. | $\int dn$. | $\int d\varpi$. | $\int d\zeta$. |
|---|---|---|---|---|
| | ° ° | ″ | ″ | ″ |
| Venus | 0– 30 | + ·019 | ... | + 532 |
| ,, | 330–360 | − ·0035 | ... | ... |
| Earth | 0– 30 | + ·007 | ... | + 196 |
| ,, | 330–360 | − ·013 | ... | ... |
| Jupiter | 0– 90 | + ·4535 | −531 | +12807 |
| ,, | 90–270 | − ·2780 | −203 | − 2905 |
| ,, | 270–360 | −1·2940 | +708 | + 22 |
| Saturn | 0– 90 | − ·0560 | − 23 | − 1578 |
| ,, | 90–270 | + ·0754 | − 6 | + 1979 |
| ,, | 270–360 | − ·2334 | + 7 | + 47 |
| Uranus | 0–360 | + ·0030 | ... | + 55 |
| | Sums | −1·3200 | − 48 | +11155 |